A SELF-SIMILAR SOLUTION OF THE BOUNDARY LAYER EQUATIONS

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We consider one of the cases in which a solution of the Falkner–Skan equation and its axially-symmetric analog is obtainable in closed form.

1. In the case of a stationary laminar boundary layer of an incompressible fluid with a power-law velocity distribution $U = cx^m$ at the outer boundary of the layer, with velocity components u and v given by expressions of the form

$$u = cx^m f'(\eta), \quad v = -\sqrt{\nu c\alpha}\, x^\gamma \left(f + \frac{\gamma}{\alpha}\eta f'\right), \qquad (1.1)$$
$$\eta = \sqrt{\alpha c/\nu}\, y x^\gamma, \ \alpha = (m+1)/2, \ \gamma = (m-1)/2$$

we have the well-known Falkner–Skan equation [1]

$$f''' + ff'' = \beta(f'^2 - 1), \ \beta = 2m/(m+1) \qquad (1.2)$$

with boundary conditions

$$f = f_0, \ f' = 0(\eta = 0), \ f' = 1(\eta = \infty), \qquad (1.3)$$

where f_0 is associated with the rate v_0 of suction ($f_0 > 0$) or blowing ($f_0 < 0$) through the surface: $v_0 = -\sqrt{\nu c\alpha}x^\gamma f_0$.

Next we consider the case $m = -1/3$ in which Eq. (1.2) may be integrated twice directly:

$$f' + (f^2 - \eta^2)/2 = A\eta + B \qquad (1.4)$$

(A and B are integration constants). By the change of variables

$$f = (\eta + A)\left[2\frac{w'(z)}{w(z)} - 1\right], \quad z = (\eta + A)^2/2 \qquad (1.5)$$

Eq. (1.4) is transformed into the degenerate hypergeometric equation

$$zw'' + (b - z)w' - aw = 0; \qquad (1.6)$$

$$a = (1/4)(1 + B - A^2/2), \ b = 1/2, \qquad (1.7)$$

the general solution of which is of the form [2]

$$w = c_1 M(a, b, z) + c_2 U(a, b, z). \qquad (1.8)$$

Here c_1 and c_2 are arbitrary constants, with the solution (1.5) dependent only on their ratio $H = c_1/c_2$.

The three arbitrary constants A, B, H, contained in the solution (1.5), (1.7), (1.8), must, generally speaking, be determined by the three boundary conditions (1.3). However, the structure of the solution is such that the condition $f' = 1$ for $\eta \to \infty$ is satisfied independently from the values of the constants A, B, H. Indeed, calculating f' from relations (1.4) and (1.5), and using the asymptotic and differential properties of the degenerate hypergeometric functions [2], we obtain

$$f' = 4z\frac{w'}{w}\left(1 - \frac{w'}{w}\right) + 4a - 1, \qquad (1.9)$$

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$$\frac{w'}{w} \approx \frac{1 - a(b-a)/z}{1 + (1-a)(b-a)/z} \approx 1 - \frac{b-a}{z}, \tag{1.9}$$

whence, with the aid of relations (1.7), it follows that $f' \approx 1$. Thus, the solution of the boundary value problem (1.2), (1.3) for $m = -1/3$ is not unique, as is also the case for other $m < 0$ [3, 4].

After determining the constants B and H from the first two of the conditions (1.3), we have a family of solutions which, for a given f_0, depend on A as a parameter. If, first of all, we single-out from this family those solutions which correspond to nonseparated flows, we find from the condition $f''(0) > 0$ that we need only consider values of $A > 0$.

Further narrowing-down the family of solutions can be effected by identifying solutions satisfying the requirement of monotonicity of increase of the longitudinal component of the velocity [the quantity $f'(\eta)$, see Eq. (1.1)] across the boundary layer cross section. A necessary (but not sufficient) condition of monotonicity is that $f' \to 1$ as $\eta \to \infty$ from the side of values of $f' < 1$. To apply this condition, we calculate f' using asymptotic expressions for the degenerate hypergeometric functions, with the following [in comparison with relation (1.9)] terms in z^{-1} being taken into account:

$$f' \approx 1 + \left[w \frac{\Gamma(a)}{\Gamma(b)} z^{b-a} \exp(-z)\right]^{-2} \frac{1-2a}{z}. \tag{1.10}$$

Here $\Gamma(x)$ is the gamma-function. From this expression we obtain, upon taking relations (1.3), (1.4), and (1.7) into account, the necessary condition of monotonicity in the form $(1/4)(1 + f_0^2/2 - A^2/2) \geq 1/2$, from which it follows that solutions with monotonic variation of velocity across the boundary layer can only exist when $|f_0| \geq 2$. There are no solutions of this kind for flow over a nonpermeable surface ($f_0 = 0$), which is in accord with a theorem proved in [4]. The monotonicity condition obtained above does not permit us to consider the cases $f_0 > 0$ and $f_0 < 0$ separately; however, the solution for these cases is of a different nature since f_0 also appears in the expression for the constant H. Numerical tabulation of the solution (1.5), (1.7), (1.8) shows that solutions with $f_0 < 0$ (blowing through the surface) correspond to a nonmonotonic variation of f': as η increases, it passes through a maximum in the region $f' > 1$ and through a minimum in the region $f' < 1$; only after this does it approach $f' = 1$.

When $f_0 > 0$ and the values of A lie in the interval $0 < A \leq \sqrt{f_0^2 - 2}$, the variation of $f'(\eta)$ is monotonic for all solutions satisfying relations (1.3), (1.4), and in singling-out a unique solution it is necessary to use a criterion first introduced in [3], namely, that one picks out a solution having the largest rate of approach of f' to one as $\eta \to \infty$. In [4] another formulation of the principle of identifying a unique solution is given, one in which the "free" constant is determined at the expense of posing the boundary condition $f' = 1$ for some finite $\eta = k$, followed by a passage to the limit $k \to \infty$ in the expressions thus obtained; this also results in singling-out a solution with largest rate of approach of f' to one. In [5] a justification of this criterion is given in which a study is made of the stability of a self-similar solution with respect to nonself-similar perturbations.

It follows from relation (1.10) that the solution with the largest rate of approach of f' to one as $\eta \to \infty$ corresponds to $a = 1/2$ (or $A = \sqrt{f_0^2 - 2}$). Numerical tabulation confirms this result. Since $a = b = 1/2$ Eq. (1.6) is solvable in terms of quadratures:

$$w = \exp(z)\left[c_1 + c_2 \int z^{-1/2} \exp(-z)\, dz\right],$$

the solution of problem (1.2), (1.3) for $m = -1/3$ ($\beta = -1$) can be written in closed form:

$$f = \tau\sqrt{2}(g+1),\ f' = 1 - \tau^2 g(g+2),\ \tau = (\eta + A)/\sqrt{2},$$

$$g(\tau) = \tau^{-1}\exp(-\tau^2)\left[H + \frac{\sqrt{\pi}}{2}\operatorname{erf}(\tau)\right]^{-1},$$

$$H = \frac{\sqrt{2}\exp(-A^2/2)}{f_0 - A} - \frac{\sqrt{\pi}}{2}\operatorname{erf}(A/\sqrt{2}),\ A = \sqrt{f_0^2 - 2}.$$

Here $\operatorname{erf}(x)$ is the probability integral. This solution exists only for $f_0 \geq 2$, which confirms a theorem in [6] according to which a unique solution of problem (1.2), (1.3) for $\beta < -0.199$, singled-out on the basis of the criterion in [3], exists for a suction rate exceeding a critical value.

2. We consider an analog of the Falkner–Skan equation for axially-symmetric boundary layers, generated in flow longitudinally over thin bodies of revolution. Body elongation is assumed to be fairly large: the angle between the tangent to a meridian curve of the surface and the axis of the body is a small quantity. For thin bodies, where the boundary layer thickness is comparable, in order of magnitude, with the radius of curvature of the body, the boundary layer equations have the form [7]

$$v_x \frac{\partial v_x}{\partial x} + v_r \frac{\partial v_x}{\partial r} = UU' + \nu\left(\frac{\partial^2 v_x}{\partial r^2} + \frac{1}{r}\frac{\partial v_x}{\partial r}\right),\qquad \frac{\partial v_x}{\partial x} + \frac{\partial v_r}{\partial r} + \frac{v_r}{r} = 0, \tag{2.1}$$

where x and r are cylindrical coordinates (the x-axis coincides with the axis of the body).

Writing a self-similar solution of Eqs. (2.1) as

$$v_x = cx^m f'(\eta),\ v_r = -\frac{2\nu}{r}[f + (m-1)\eta f'],\ \eta = \frac{cr^2}{4\nu x^{1-m}}, \tag{2.2}$$

we arrive at the boundary value problem

$$mf'^2 - ff'' = m + (\eta f'')'; \tag{2.3}$$

$$f = f_0,\ f' = 0(\eta = \eta_0),\ f' = 1(\eta = \infty),\qquad \eta_0 = ca^2/(4\nu), \tag{2.4}$$

corresponding to a power-law velocity distribution $U = cx^m$ on the outer boundary of the layer and a given shape $r_0 = ax^{(1-m)/2}$ for the surface of the body.

We remark that the given combination of functions U(x) and $r_0(x)$ is stipulated only by the requirement of self-similarity of the solution and is not connected with the laws of nonviscous flow. As a result of the smallness of the angle of inclination of the surface of the body to its axis, the longitudinal component of the velocity of the potential flow close to the surface (at the outer edge of the layer) coincides approximately with the velocity of the potential flow at infinity. Therefore, the velocity distribution U(x) can be assumed to be given independently of the shape $r_0(x)$ of the surface. In particular, the case $m = -1$, considered next, can be treated as a boundary layer on a thin cone ($r_0 = ax$, $a \ll 1$), placed inside a flat diffuser ($U = c/x$).

When $m = -1$, Eq. (2.3), after two integrations, leads to the first order equation

$$\eta f' + (f^2 - \eta^2)/2 - f + A\eta + B = 0 \tag{2.5}$$

(A and B are constants). By means of the substitution

$$f = b - \eta + 2\eta\frac{w'(\eta)}{w(\eta)},\ b = 1 \pm \sqrt{1-2B} \tag{2.6}$$

this equation is reduced to the degenerate hypergeometric equation

$$\eta w'' + (b - \eta)w' - aw = 0,\ a = (b - A)/2; \tag{2.7}$$

$$w = c_1 M(a, b, \eta) + c_2 U(a, b, \eta). \tag{2.8}$$

From the formulas of transformation of hypergeometric functions it follows that f is independent of the choice of sign in the expression (2.6) for b.

Using asymptotic expansions for $M(a, b, \eta)$ and $U(a, b, \eta)$, we can, as was done in Sec. 1, show that the solution (2.6)-(2.8) satisfies the boundary condition at infinity independently of the values of the constants A, B, and $H = c_1/c_2$:

$$f' \approx 1 + \left[w \frac{\Gamma(a)}{\Gamma(b)} z^{b-a} \exp(-z)\right]^{-2} (b-a)(1-a)/\eta^2. \tag{2.9}$$

Determining the constants B and H from the first two of the conditions (2.4), we obtain a family of solutions depending on η_0, f_0, and A, where the parameters a and b of the hypergeometric functions are represented, after elimination of B, in the following way:

$$b = 1 \pm \sqrt{(f_0 - 1)^2 + \eta_0(2A - \eta_0)}, \quad a = (b - A)/2. \tag{2.10}$$

In identifying a unique solution we use the very same principles as were used in Sec. 1. The condition for nonseparation of the flow and use of relations (2.4), (2.5) yields $A < \eta_0$. A solution satisfying the necessary condition of monotonicity of the distribution $f'(\eta)$ and the corresponding condition of largest rate of approach of f' to one as $\eta \to \infty$ may be obtained on the basis of relation (2.9) for $a = b$ or for $a = 1$. An analysis using formulas (2.10) shows that these two cases turn into one another upon interchanging the sign in the expression for b and lead to one and the same equation for determining the two possible values for A:

$$A = \eta_0 - 1 \pm \sqrt{(f_0 - 1)^2 - 2\eta_0}. \tag{2.11}$$

A numerical tabulation of the solution shows that here it is necessary to choose the lower sign since this value of A corresponds to a more rapidly increasing solution. The condition $A < \eta_0$ for nonseparation of the flow is thereby satisfied for arbitrary η_0 and f_0.

It also follows from the results of the numerical tabulation that monotonicity in the variation of the longitudinal velocity across the layer only holds when the condition that A be real in Eq. (2.11) is chosen in the form $f_0 \geq 1 + \sqrt{2\eta_0}$; in the contrary case, the function $f'(\eta)$ goes through a maximum and a minimum before reaching a stationary value at infinity.

The unique solution of problem (2.4), (2.5), identified in this way, can be represented in closed form ($a = b = -A$)

$$f = -A + \eta(G + 1), \; f' = 1 + G(1 + A - \eta) - \eta G^2/2,$$
$$G(\eta) = 2\eta^A \exp(-\eta)[H + K(\eta)]^{-1},$$
$$H = \frac{2\eta_0^{A+1}\exp(-\eta_0)}{f_0 + A - \eta_0} - K(\eta_0), \quad K(\eta) = \int_0^\eta t^A \exp(-t)\, dt,$$
$$A = \eta_0 - 1 - \sqrt{(f_0 - 1)^2 - 2\eta_0}.$$

For discrete positive values of $A = n$ (for discrete values of suction intensity $f_0 = 1 + \sqrt{(n+1)^2 + \eta_0^2 - 2\eta_0 n}$, providing $\eta_0 > n$) the solution may be written in terms of elementary functions:

$$K(\eta) = -\exp(-\eta)\left[\eta^n + \sum_{l=1}^{n} \frac{n!}{(n-l)!}\eta^{n-l}\right].$$

For discrete negative values of $A = -n$ (discrete $f_0 = 1 + \sqrt{(n-1)^2 + \eta_0^2 + 2\eta_0 n}$ for arbitrary η_0) the expression for $K(\eta)$ contains an integral exponential function. For half-integer values of A we can express $K(\eta)$ in terms of the probability integral.

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DISTURBANCES OF HIGH MODES IN A SUPERSONIC JET

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This article analyzes and reports the results of numerical modeling of a little-studied wavelike phenomenon seen in jet flows. Alternating light and dark bands are visible on shadowgraphs of supersonic jets flowing from circular nozzles in the underexpansion regime. These bands, present on the initial section (the first and part of the second zones), are evidence of the presence of azimuthal flow irregularities in these regions. Figure 1 [1] shows a typical Topler photograph illustrating the above. Such patterns are familiar in gasdynamics, although no explanations for them are offered either in this well-known work or in monographs on jets. Two recent experimental studies have returned to this question, the authors using different approaches in each case. The method laser diagnostics was used in [2] to study the disturbance of azimuthal symmetry for a jet discharged into a vacuum in the preturbulent regime. It was shown that the compressed layer breaks up into a certain number of lobes which are interspersed with gas from the surrounding space. This results in the formation of different transverse distributions of density ρ at different azimuthal angles φ. It was suggested such an event might be a consequence of the onset of instability in the flow. The authors of [1] probed the region of the compressed layer between a suspended shock wave and the boundary of a submerged turbulent jet by traditional methods used to measure gasdynamic quantities — with a pilot tube inserted into the flow and positioned coaxially with possible streamlines. The resulting variations in total pressure indicate the existence of azimuthal irregularities of the longitudinal-velocity distributions in the region of the compressed layer. The authors stated that this is a consequence of the presence of longitudinal vorticity of the Taylor—Görtler vortex type in the flow.

The hypothesis on the wavelike nature of the observed bands is supported by experimental data obtained in the related areas of internal gasdynamics, aerodynamics, and the hydrodynamic stability of boundary flows. Here it has been possible to use visualization methods and to reliably identify the alternating bands with eddies. The genesis of these eddies may be different, however. Coherent structures in the form of stationary longitudinal vortices which are periodic with respect to the transverse coordinate (Benny—Lin vortices [3, 4]) are widely known in hydrodynamic stability. They are formed as a result of synergetic processes — the spontaneous formation of structures of a certain type of instability wave with a finite intensity. Longitudinal eddies connected with curvature of the streamlines are observed on the inside surfaces of nozzles [5], in boundary layers on concave surfaces [6], and in separated flows at sites of flow attachment [7]. Without stopping to analyze these flows in detail, it is necessary to emphasize that the turbulence mechanism noted above may also be operative in the underexpanded jet being examined here.

Thus, the initial motivation for the present investigation was to check the hypothesis of the possible existence of longitudinal vorticity in free supersonic jets in the form of stationary eddies located in the region of the compressed layer and oriented with the flow. Since the modeling will be done with incomplete information (little experimental data) and since it will therefore be impossible to unambiguously establish the origin of the vorticity, we will examine structures whose origin is related to unstable oscillations of various types.

First of all, these are waves which are steady over time and are connected with curvature of the trajectories of the gas due to the intrinsic shock-wave configuration of the

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